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The Initial Value Problem for Fractional
Order Differential
Equations with Constant Coefficients

by

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Abstract

Systems of fractional order differential equations are constructed to solve for viscoelastically damped structural motion. To reduce computational effort the equations of motion are cast as an initial value problem where the history dependence is approximated by retaining events only from the recent past. Several strong parallels with ordinary, linear differential equations arise. These parallels plus the well-posed nature of the fractional order differential equations leads one to view the fractional order initial value problem as an extension of the theory of ordinary, linear differential equations with constant coefficients.

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## Introduction

This work focuses on constructing and solving the initial value problem for systems of fractional order differential equations. These equations arise when fractional order derivatives are used to describe the linear viscoelastic behavior of elastomeric materials in the equations of motion of damped structures (1)(3)(5). The fractional order derivative is particularly well suited to describe the frequency dependent stiffness observed in many viscoelastic materials (2)(6)(7)(8)(9)(20). These viscoelastic stress-strain constitutive relationships are constructed by replacing the ordinary derivatives appearing in the classical linear viscoelastic model (11) with fractional order derivatives (6)(8)(14)(15)(18).

Because of their ability to describe weak frequency dependence, significantly fewer fractional derivative terms are needed to relate time-dependent stress and strain fields. Typically models with four or five parameters effectively describe the material stiffness over five or more decades of frequency. The small number of parameters makes least squares fitting of the data attractive and several materials have been modeled (3)(4)(6)(20). These constitutive models are empirical, but their general mathematical form is suggested by approximate molecular theories that relate microstructure to macro viscoelastic stiffness properties (4).

More important to the engineer, these models can be straightforwardly incorporated into continuum and finite element formulations for the analysis of structures containing in principle and unlimited number of linear elastic and linear

viscoelastic materials (1). These formulations lead to closed form, real, continuous and causal solutions using Laplace transforms or Green's function solutions and convolution (1)(3). The fractional calculus model that produces these solutions, like most viscoelastic models, is a hereditary operator that requires a complete displacement or strain history to produce precise results. Present solution techniques are predicated on a total history (1)(3)(5).

Fortunately, the fractional calculus model is also a fading memory operator where events in the distant past have less effect on the present and future states than do comparable events in the recent past. This feature raises the possibility of ignoring events in the distant past and generating approximate structural responses based the recent past (a few characteristic times back in history), the present state and future loadings. When the effects of the recent past are cast as pseudo forcing functions superimposed on the future loads, one has constructed an initial value problem where the structural response is uniquely dependent on the present state and future loads. Solving the initial value problem, instead of calculating the response based on a complete displacement history, clearly reduces the effort required to produce solutions and obviates the need to begin the analysis in the distant past at which time the viscoelastic materials were in a virgin, quiescent state.

The initial value problem is posed as a system of integro-differential equations, which upon close examination may be viewed as differential equations generalized to fractional order. These equations are referred to as fractional order

differential equations with some justification because of their similarity with ordinary differential equations with constant coefficients. For instance, it is shown that a system of m fractional order differential equations produces m eigenfunctions needing m initial conditions for a unique solution. These eigenfunctions will be cast in terms of Mittag-Leffler functions (16), long viewed by some as generalized exponential functions (12:280)(15:260)(18:44). The total solution for a fractional order differential equation is seen to be composed of a generalized homogeneous solution uniquely dependent on the initial value and a generalized particular solution uniquely dependent on the forcing function.

Setting the fractional order to integer values produces ordinary differential equations with constant coefficients (13:527) and the generalized solutions become the traditional solutions to systems of ordinary differential equations. In addition it is shown that the fractional order differential equations produce non-singular Green's functions which insure continuous dependence on the data which, when coupled with existence and uniqueness considerations, lead to well-posed problems. The preponderence of the strong parallels between fractional order differential equations and ordinary differential equations lends credence to the view that this practical tool for the engineer is in fact a generalization to fractional order of the theory of ordinary, linear, differential equations with constant coefficients.

Constructing the Fractional Order Differential Equations of Motion

Differential equations of fractional order arise in solid mechanics when rational fractional order derivatives are employed to describe the viscoelastic phenomenon. This approach produces fractional order differential equations of motion where the elastic constants are replaced by fractional derivative operators intended to describe the time-dependent viscoelastic stress-strain moduli. The extended Riemann Liouville fractional derivative[1] (17:50)(19:19) is a linear operator and is defined as

$$D^{\alpha}_{(t)}[u(t)] \equiv \frac{d}{dt}\int_0^t \frac{u(t)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}\,d\tau \qquad 0 \le \alpha \le 1 \qquad (1)$$

This operator is used to construct viscoelastic stress-strain constitutive relationships of the form

$$\sigma(t,x_i) + \sum_{p=1}^{N} b_p D^{\alpha_p}_{(t)}[\sigma(t,x_i)] = E_o\epsilon(t,x_i) + \sum_{p=1}^{N} E_p D^{\alpha_p}_{(t)}[\epsilon(t,x_i)] \qquad (2)$$

shown here for uniaxial deformation. This constitutive relationship may be viewed as a generalization to fractional order (1:17) of the classical viscoelastic model (11:14) where derivatives of integer order are used to relate empirically time-dependent stress and strain fields.

This fractional derivative constitutive relationship has been successfully incorporated into the equations of motion for continuum and finite element formulations producing real,

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[1] This definition is usually presented with $\alpha$ restricted to less than one. However, Güttinger (13:527) has proven that the weak limit of the kernel as $\alpha$ tends to one is the Dirac delta function, in which case the definition produces a first derivative.

continuous and causal solutions for forced structural response. The solution method for the resulting equations of motion is based on decomposing them into systems of decoupled linear, fractional order differential equations (3)(5).

The overall approach may be demonstrated by constructing the equation for forced motion of a simple spring-mass harmonic oscillator and generalizing the result for multiple degree of freedom structures having viscoelastic components.

$$F(t) - M\ddot{u}(t) = ku(t) \qquad (3)$$

Modeling the spring as a massless bar (with elastic modulus E, cross-sectional area A and length L) yields

$$\sigma(t) = \frac{F(t) - M\ddot{u}(t)}{A} = \frac{Eu(t)}{L} = E\epsilon(t) \qquad (4)$$

This may be viewed as an elastic stress-strain relationship where the inertially induced stress is described using D'Alembert's principle. The massless rod is now taken to be viscoelastic and modeled using eqn 2, where the elastic terms and the first fractional derivative terms acting on stress and strain are retained.

$$(1 + bD^{\alpha})\sigma(t) = (E_o + E_1 D^{\alpha})\epsilon(t) \qquad (5)$$

The resulting equation of motion in operator form is

$$(1 + bD^{\alpha})\left[\frac{F(t) - M\ D^2 u(t)}{A}\right] = (E_o + E_1 D^{\alpha})\left[\frac{u(t)}{L}\right] \ . \qquad (6)$$

Using the composition property (17:30) of the fractional derivative operator,

$$D^{\alpha}\left[D^{\gamma}\left[u(t)\right]\right] = D^{\alpha+\gamma}\left[u(t)\right], \qquad (7)$$

the equation of motion takes the form

$$b\underline{\underline{M}}D^{2+\alpha}\underline{u}(t) + \underline{\underline{M}}D^{2}\underline{u}(t) + \underline{\underline{k}}_1 D^{\alpha}\underline{u}(t) + \underline{\underline{k}}_o\underline{u}(t) = bD^{\alpha}\underline{F}(t) + \underline{F}(t). \qquad (8)$$

Here the equations of motion have been expanded to describe the behavior of systems with several degrees of freedom. The double and single underlined variables are square and column matrices, respectively, of order equal to the number of physical degrees of freedom, N. Identifying the largest factor $\beta$, of the form 1/n, common to all the fractional orders of differentiation[2] in eqn 8 and again applying the composition property, this equation of motion becomes

$$\left(b\underline{\underline{M}}(D^{\beta})^{m} + \underline{\underline{M}}(D^{\beta})^{w} + \underline{\underline{k}}_1(D^{\beta})^{q} + \underline{\underline{k}}_o\right)\underline{u}(t) = \left(1 + b(D^{\beta})^{q}\right)\underline{F}(t) \qquad (9)$$

where m, w and q are integers and $(D^{\beta})^{m}$ is the $\beta$ order derivative taken m times. The orders of the corresponding differential operators in eqns 8 and 9 are also equal.

$$\begin{aligned} \beta m &= 2 + \alpha \\ \beta w &= 2 \\ \beta q &= \alpha \\ \beta &= 1/n \end{aligned} \qquad (10)$$

The most general form of these equations of motion is

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[2] This restriction on $\beta$ is tied to solving the initial value problem. When solving for the motion starting with zero displacement and velocity, this restriction is not needed and $\beta$ may be taken to be the largest rational factor common to all orders of differentiation.

$$\sum_{p=0}^{m} \underline{\underline{c}}_p (D^{\beta})^p \, \underline{u}(t) = (1 + b(D^{\beta})^q)\underline{F}(t) = \underline{f}^*(t). \qquad (11)$$

Here the $\underline{\underline{c}}_p$ are real, although many may be zero, and $\underline{f}^*(t)$ is the result of the viscoelastic stress operator acting on the applied forces, $\underline{F}(t)$, as shown in eqn 9.

Eqn 11 poses the system as N equations of order $\beta$m that can be alternatively posed as m·N equations of order $\beta$. In matrix form the m·N equations of $\beta$ order are

$$D^{\beta}\left[\begin{array}{c:ccc} \underline{\underline{0}} & & & \\ \vdots & & A & \\ \underline{\underline{0}} & & & \\ \underline{\underline{0}} & & & \\ \hdashline \underline{\underline{c}}_m & \cdots \underline{\underline{c}}_3 & \underline{\underline{c}}_2 & \underline{\underline{c}}_1 \end{array}\right] \begin{Bmatrix} (D^{\beta})^{m-1}\underline{u}(t) \\ \vdots \\ (D^{\beta})^{2}\underline{u}(t) \\ (D^{\beta})^{1}\underline{u}(t) \\ \underline{u}(t) \end{Bmatrix} \qquad (12)$$

$$+ \left[\begin{array}{ccc:c} & & & \underline{\underline{0}} \\ & -A & & \vdots \\ & & & \underline{\underline{0}} \\ & & & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & \cdots & \underline{\underline{0}} \; \underline{\underline{0}} & \underline{\underline{c}}_0 \end{array}\right] \begin{Bmatrix} (D^{\beta})^{m-1}\underline{u}(t) \\ \vdots \\ (D^{\beta})^{2}\underline{u}(t) \\ (D^{\beta})^{1}\underline{u}(t) \\ \underline{u}(t) \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \underline{f}^*(t) \end{Bmatrix}$$

where the lowest equation is seen to be eqn 11. The matrix [A] is chosen such that both square matrices of order m·N become symmetric and the top (m-1)·N equations are satisfied identically. This is accomplished by constructing A such that all matrices, $\underline{\underline{c}}_p$ lying on any given diagonal running from lower left to upper right in the first matrix of eqn 12, are equal. This form of the equations of motion will be referred to as the expanded equations of motion.

For example, if $\alpha$ is one half in eqn 8, then $\beta$ is one half

making m=5 in eqn 9, and the expanded equations of motion become

$$D^{1/2}\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & b\underline{\underline{M}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & b\underline{\underline{M}} & \underline{\underline{M}} \\ \underline{\underline{0}} & \underline{\underline{0}} & b\underline{\underline{M}} & \underline{\underline{M}} & \underline{\underline{0}} \\ \underline{\underline{0}} & b\underline{\underline{M}} & \underline{\underline{M}} & \underline{\underline{0}} & \underline{\underline{0}} \\ b\underline{\underline{M}} & \underline{\underline{M}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{k}}_1 \end{bmatrix} \begin{Bmatrix} (D^{1/2})^4\underline{u}(t) \\ (D^{1/2})^3\underline{u}(t) \\ (D^{1/2})^2\underline{u}(t) \\ (D^{1/2})^1\underline{u}(t) \\ \underline{u}(t) \end{Bmatrix}$$

$$+\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & -b\underline{\underline{M}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & -b\underline{\underline{M}} & -\underline{\underline{M}} & \underline{\underline{0}} \\ \underline{\underline{0}} & -b\underline{\underline{M}} & -\underline{\underline{M}} & \underline{\underline{0}} & \underline{\underline{0}} \\ -b\underline{\underline{M}} & -\underline{\underline{M}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{k}}_0 \end{bmatrix} \begin{Bmatrix} (D^{1/2})^4\underline{u}(t) \\ (D^{1/2})^3\underline{u}(t) \\ (D^{1/2})^2\underline{u}(t) \\ (D^{1/2})^1\underline{u}(t) \\ \underline{u}(t) \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ (1+bD^{1/2})\underline{F}(t) \end{Bmatrix} = \left\{ f^*(t) \right\} \tag{13}$$

Both the general form, (eqn 12), and the example in eqn 13 are now posed in terms of two real, square, symmetric matrices for which an orthonormal transformation exists

$$\begin{Bmatrix} (D^{\beta})^{m-1}\underline{u}(t) \\ \vdots \\ (D^{\beta})^2\underline{u}(t) \\ (D^{\beta})^1\underline{u}(t) \\ \underline{u}(t) \end{Bmatrix} = \begin{bmatrix} \Phi \end{bmatrix} \begin{Bmatrix} y_{m\cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{Bmatrix} \tag{14}$$

that leads to a system of m·N uncoupled differential equations of order $\beta$.

$$D^{\beta}\begin{bmatrix} \diagdown_{I}\diagdown \end{bmatrix}\left\{y(t)\right\} + \begin{bmatrix} \diagdown_{a}\diagdown \end{bmatrix}\left\{y(t)\right\} = \begin{bmatrix} \phi \end{bmatrix}^T\left\{ f^*(t)\right\} = \left\{f(t)\right\} \tag{15}$$

Moreover, one can see that this process is analogous to posing systems of higher order coupled ordinary differential equations with constant coefficients as a collection of first order differential equations, the difference being that fractional order differential equations result here.

At this point one might ask why we have resorted to this formalism. In general, the stiffness matrices $\underline{\underline{k}}_1$ and $\underline{\underline{k}}_o$ are distinctly different matrices when two or more materials are present in the structure and at least one is viscoelastic. Constructing an orthonormal transformation that decouples a system with more than two (in this case two stiffness and one mass) symmetric matrices is generally not possible. Consequently, casting this system of equations in terms of two symmetric matrices, eqn 12, becomes necessary to construct manageable decoupled equations through an orthonormal transformation.

The major drawback of having to rely on the formalisms of eqns 12 and 14 is that they require the manipulation of large, usually sparsely populated, matrices. Numerical methods, that capitalize on the repetitive structure of eqn 12 and the less obvious repetitive structure of eqn 14 to increase efficiency, have been applied to extract eigenvalues and eigenvectors for a 65 degree of freedom system with only modest computational effort (5:923). When using fractional derivative models to determine the response of structures containing viscoelastic components, the formalism of eqn 12 is necessary, if decoupled equations of motion are required for modal analyses. In any event, the fractional order basis equations shown in eqn 15 are the progenitor of the initial value problem.

Constructing the Initial Value Problem

The decoupled fractional order equations of motion or basis equations, (eqn 15) individually take the form

$$(D^{\beta} + a^{\beta})y(t) = f(t) \qquad \beta = 1/n \qquad (16)$$

where the subscripts have been dropped to simplify notation. Green's function solutions for these equations are relatively straightforward and the resulting expressions for the forced response of the structure can be shown to be real, continuous and causal (1:73) These solutions to eqn 16 may be viewed as particular solutions of the fractional order differential equation of motion.

It is curious to note that the only homogeneous solution to eqn 16 is the trivial solution. This appears to be consistent with a strict interpretation of eqn 2, the generalized viscoelastic constitutive model. Inherent to the model is the implication that the relationship between stress and strain at any given time should be a function of the complete stress and strain histories. In other words, at time zero the viscoelastic material should be in its virgin, undeformed state where structural motion is commencing from a quiescent state. Attempting to impose non-trivial initial conditions implies the existence of previous motion that is inconsistent with the viscoelastic model and, therefore, homogeneous solutions are not needed.

In practice one usually does not have, cannot calculate or should not calculate a complete time history of structural motion to conform with this restriction. For example consider the case

of a viscoelastically damped structure subjected to several episodic loadings, where time intervals between episodes do not allow the structure to assume an essentially undeformed, quiescent state. As a practical matter, one would like to be able to determine the structure's response for one loading episode where its initial displacements and velocities are significant and known, and a "recent" history of structural motion is available.

To produce equations of motion which satisfy this need, eqn 16 will be posed in terms of a shifted time scale, $\tilde{t}$, shown in Figure 1.

$$\frac{1}{\Gamma(1-\beta)} \frac{d}{d\tilde{t}} \int_{0}^{\tilde{t}} \frac{y(\tau-t_o)}{(\tilde{t}-\tau)^{\beta}} d\tau + a^{\beta} y(\tilde{t}-t_o) = f(\tilde{t}-t_o) \qquad (17)$$

Here $t_o$ is the time of the onset of the loading history for which the response is needed. Time zero on the $\tilde{t}$ timescale is taken to be several characteristic times, $a^{-1}$, preceding the beginning of this load. This preceding time interval is not restricted to being load free and in general may have loading present right up to the start of the loading history of interest. The loads prior to $t_o$ are $\tilde{f}\ (\tilde{t}-t_o)$ and the corresponding response is $\tilde{y}(\tilde{t}-t_o)$. The equation of motion resulting from $\tilde{f}(\tilde{t}-t_o)$ is

$$D^{\beta}\tilde{y}(\tilde{t}-t_o) + a^{\beta}\tilde{y}(\tilde{t}-t_o) = \tilde{f}(\tilde{t}-t_o) \qquad (18)$$

The loads for the episode of interest ($\tilde{t} \geq t_o$) are $\tilde{\tilde{f}}(\tilde{t}-t_o)$ and the equation for the corresponding response $\tilde{\tilde{y}}(\tilde{t}-t\ )$ is

$$D^{\beta}\tilde{\tilde{y}}(\tilde{t}-t_o) + a^{\beta}\tilde{\tilde{y}}(\tilde{t}-t_o) = \tilde{\tilde{f}}(\tilde{t}-t_o) \qquad (19)$$

The total response for $t \geq t_o$ is $\tilde{\tilde{y}} + \tilde{y}$ and the general

expression for the response is

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\tilde{\tilde{y}}'(r-u) + \tilde{y}'(r-u)}{u^\beta} du + a^\beta\left(\tilde{\tilde{y}}(r) + \tilde{y}(r)\right)$$

$$= \tilde{\tilde{f}}(r) + \tilde{g}(r) \tag{20}$$

where $r = \tilde{t}-t_o$, $u = \tilde{t}-\tau$. Here $\tilde{g}(r)$ is a pseudo forcing function that produces the residual response of the structure due to the prior application of $\tilde{f}(t-t_o)$.

$$\tilde{g}(r) = -\frac{1}{\Gamma(1-\beta)}\left\{\int_r^{r+t_o} \frac{\tilde{y}'(r-u)}{u^\beta} du + \frac{\tilde{y}(-t_o)}{(r+t_o)^\beta}\right\} \tag{21}$$

Expressing eqn 20 in terms of the t time scale in Figure 1, where zero time is now the onset of the loading episode of interest, yields

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{y'(t-\tau)}{\tau^\beta} d\tau + a^\beta y(t) = \tilde{\tilde{f}}(t) + \tilde{g}(t) = g(t). \tag{22}$$

Note that here the order of differentiation and integration in the fractional derivative operator is the opposite of eqn 1. This reversal of operations occurred when Leibnitz's rule was used to differentiate the integral in eqn 17, producing eqns 20 and 21. This change will prove crucial to solving the initial value problem, because in contrast with eqn 16, eqn 22 possesses both a particular solution, uniquely dependent on the forcing function, and a homogeneous solution, uniquely dependent on the initial value, y(0).

Before presenting these solutions it is important to address

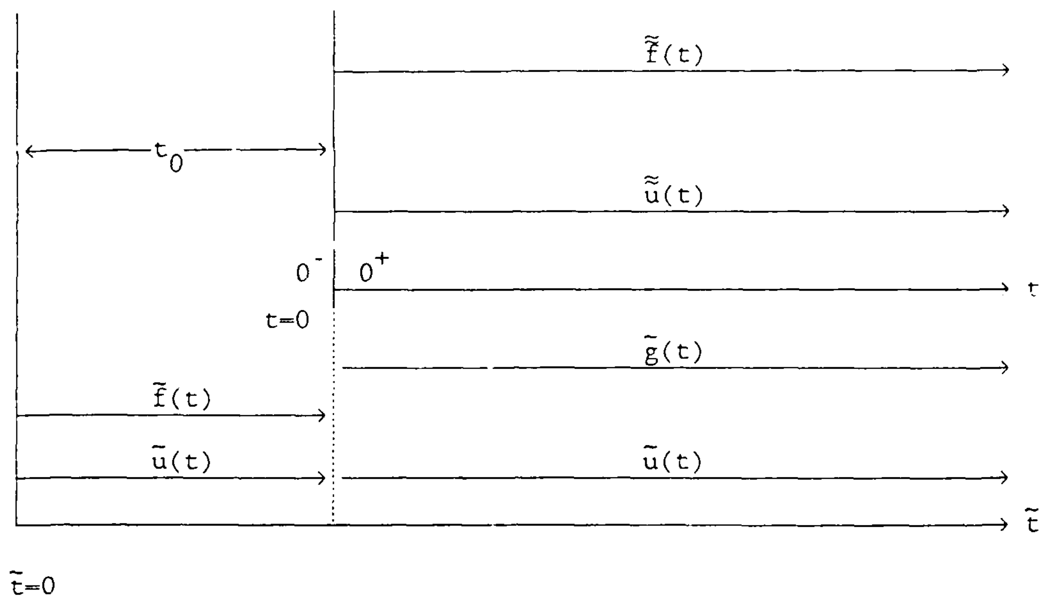


Figure 1 - Time Scales for the Loads and Responses of the Initial Value Problem.

the relationship between the operator appearing in eqn 22 and the original definition shown in eqn 1. Using Leibnitz's rule to differentiate the integral in eqn 1 yields

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(t-\tau)}{\tau^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{u(0)}{t^\alpha} + \int_0^t \frac{u'(t-\tau)}{\tau^\alpha} d\tau \right\} \qquad (23)$$

or in operator form

$$D^\alpha\left[u(t)\right] = \frac{u(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \hat{D}^\alpha\left[u(t)\right] \qquad (24)$$

where $D^\alpha$ is the definition and $\hat{D}^\alpha$ is the modified operator appearing in eqn 22. In fact $\hat{D}^\alpha$ is the Riemann-Liouville fractional integral of order $1-\alpha$ of the first derivative of the function or effectively an order $-\alpha$ integal of a function.[3] In effect the operator $D^\alpha$ treats u(t) as though it is zero for negative time and is "turned on" by a step function at time zero. The singular term appearing in eqns 23 and 24 is the $\alpha$ order time derivative of a step function with magnitude u(0). Conversely, the $\hat{D}^\alpha$ operator treats u(t) as though its value for $t = 0^+$ is an analytic continuation of its non-zero value at $t = 0^-$. The existence of an analytic continuation of u(t) into positive from negative time means its first derivative is bounded at time zero. As shown in Appendix C, a bounded first derivative at $t = 0$ restricts the initial values of the modified fractional derivatives of positive, rational order less than one to zero. This result will prove crucial to the solution of the initial

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[3] The relationship between the operators (eqn 24) is developed more formally by Oldham.(17:50)

value problem.

Posing equation 22 in terms of the modified fractional derivative operator, $\hat{D}^{\beta}$,

$$(\hat{D}^{\beta} + a^{\beta})\ y(t) = \tilde{\tilde{f}}(t) + \tilde{g}(t) = g(t) \tag{25}$$

produces the modified basis equations. Note the similar appearance of eqns 16 and 25. Recall that eqn 16 is based on the $\tilde{t}$ time scale and has a trivial homogeneous solution. On the other hand, eqn 25 is based on the t time scale, possesses a non-trivial homogeneous solution and accounts for the effects of previous motion through the initial value, y(0), and pseudo forcing function, $\tilde{g}(t)$.

Solving the Initial Value Problem

The eventual goal is to use the solutions of the modified basis equations, eqn 25, to construct solutions to the original structural equations of motion for non-trivial initial displacements and velocities, where the relaxation effects induced by previous motion are accounted for by the pseudo-forcing functions, as in eqn 20. The overall homogeneous solution will be a superposition of the homogeneous solutions for the modified basis equations and will be shown to satisfy the initial conditions. The overall particular solution will be constructed from the particular solutions to the modified basis equations, derived using Green's functions. Superimposing the overall homogeneous and particular solutions produces the total solution to the initial value problem for structural motion.

The overall homogeneous solution is constructed by first solving eqn 25 for all the homogeneous solutions to the modified basis equations. These solutions take the form

$$y_h(t) = y_h(0) \sum_{p=o}^{\infty} \frac{(-(at)^{\beta})^p}{\Gamma(1+p\beta)} \tag{26}$$

which is a special case of the beta order Mittag-Leffler function defined as (16.102)

$$E_{\beta}(x) \equiv \sum_{p=o}^{\infty} \frac{(x)^p}{\Gamma(1+p\beta)} . \tag{27}$$

In Mittag-Leffler notation the homogeneous solution is

$$y_h(t) = y_h(0)\ E_{\beta}\left(-(at)^{\beta}\right), \tag{28}$$

where this special form of the Mittag-Leffler function has the property

$$\hat{D}^{\beta} E_{\beta}\left(-(at)^{\beta}\right) = -a^{\beta} E_{\beta}\left(-(at)^{\beta}\right). \tag{29}$$

Including the particular solution, the total solution to each of the modified basis equation is

$$y(t) = y_h(0)\ E_{\beta}\left(-(at)^{\beta}\right) + \int_0^t D^{1-\beta}\left[E_{\beta}\left(-(a\tau)^{\beta}\right)\right] g(t-\tau)d\tau \tag{30}$$

which can be determined using Laplace transforms or other traditional solution techniques for integral-differential equations. The kernel in the convolution integral of eqn 29 is the unit impulse solution (Green's function) for the modified basis equations, and is singular. Note that $E_{\beta}(0)$ is not zero and that the singular behavior of the kernel can be determined through a straightforward application of eqn 1.

It is the singular nature of fractional order derivatives of $E_{\beta}(-(at)^{\beta})$ that is useful in resolving an apparent paradox in the overall initial value problem. Recall that there are $m \cdot N$ (eqn 25) modified basis equations needed to characterize the structure, where the solution for each modified basis equation has a homogeneous solution containing a different initial value. For example in the single degree of freedom problem, N=1, we have only two initial conditions, displacement and velocity. It would appear as though we have insufficient information to combine m homogeneous solutions to the m modified basis equations in a unique overall homogeneous solution.

This paradox becomes more apparent when eqn 14 is used to

solve for the $m \cdot N$ initial values of the homogeneous basis functions in terms of the structure's initial displacements $\underline{u}_h(t)$ and their derivatives evaluated at time zero.

$$\left\{\begin{matrix} (D^{\beta})^{m-1}\underline{u}_h(t) \\ \vdots \\ (D^{\beta})^{2}\underline{u}_h(t) \\ (D^{\beta})^{1}\underline{u}_h(t) \\ \underline{u}_h(t) \end{matrix}\right\}_{t=0} = \left[\ \underline{\Phi}\ \right] \left\{\begin{matrix} y_{m \cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{matrix}\right\}_{t=0} \tag{31}$$

The paradox is that at this point only $\underline{u}_h(0)$ and $D^1\underline{u}_h(0)$ can be specified, while the remaining elements in the state vector on the left of eqn 31 are undetermined.

It is curious that the initial value problem should also call for the initial values of accelerations and higher order derivatives as well. Note that the order of the differential equations of motion (eqn 8) is order $2+\alpha$ or equivalently $\beta m$ and that the state vector in eqn 31 calls for the initial values of derivatives up through $2+\alpha-\beta$ or equivalently $\beta(m-1)$. In other words, when posing N, $\beta m$ order differential equations as a system of $m \cdot N$ differential equations of order $\beta$ the corresponding initial value problem calls for all the initial values of the $p\beta$ order derivatives of the displacement vector, $\underline{u}_h(t)$: $p = 0,1,2,\cdots, m-1$. These requirements appear to be analogous to the traditional initial value problem for ordinary differential equations, but also leaves one with the requirement for yet more initial conditions.

It is proven in Appendix C that all of the non-integer derivatives of $\underline{u}_h(t)$ of order less than two appearing in the state

vector have zero initial value. The initial values for acceleration and the accompanying higher order derivations appearing in the state vector can be determined by returning to the original equation of motion, eqn 8, and using successive applications, of eqn 24 to determine the singular terms in the equation of motion. The resulting equation of motion for the response to turning off the previous forcing function is

$$- b\underline{\underline{M}} \frac{\ddot{\underline{\tilde{u}}}(0^-) t^{-\alpha}}{\Gamma(1-\alpha)} - b\underline{\underline{M}} \sum_{\ell=1}^{m-2n-1} \frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)} \hat{D}^{(m-2n-\ell)\beta} \underline{u}(0^-)$$

$$+ (1 + b\hat{D}^{\alpha}) \underline{\underline{M}} \ddot{\underline{\tilde{u}}}(t) - \underline{\underline{k}}_1 \frac{\underline{\tilde{u}}(0^-) t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{\underline{k}}_o + \underline{\underline{k}}_1 \hat{D}^{\alpha}) \underline{\tilde{u}}(t) \tag{32}$$

$$= - b \frac{\underline{\tilde{F}}(0^-) t^{-\alpha}}{\Gamma(1-\alpha)} + \underline{\tilde{G}}(t)$$

The fractional derivatives in this equation of motion are evaluated for $t<0$ or equivalently $\tilde{t}<t_o$ as shown in Appendix A. $\underline{\tilde{G}}(t)$ are the pseudo forces needed to produce the residual motion associated with the previous loading history, already accounted for in the modified basis equations. The singular forcing function is the result of the $\alpha$ order derivative of the step funtion turning off $\tilde{F}(t)$. The remaining singular behavior is the result of repeatedly applying eqn 24 to separate out the singular behavior of the fractional derivatives of acceleration.

The corresponding equation of motion for the response to the new loads is

$$
\begin{aligned}
& b\underline{\underline{M}}\frac{\ddot{\underline{\tilde{\tilde{u}}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + b\underline{\underline{M}}\sum_{\ell=1}^{m-2n-1}\frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)}\hat{D}^{(m-2n-\ell)\beta}\,\ddot{\underline{\tilde{\tilde{u}}}}(0^+) \\
& + (1 + b\hat{D}^{\alpha})\underline{\underline{M}}\,\ddot{\underline{\tilde{\tilde{u}}}}(t) + \underline{\underline{k}}_1\frac{\ddot{\underline{\tilde{\tilde{u}}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{\underline{k}}_o + \underline{\underline{k}}_1\hat{D}^{\alpha})\underline{\tilde{\tilde{u}}}(t) \\
& = \frac{b\underline{\tilde{\tilde{F}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (1 + b\hat{D}^{\alpha})\underline{\tilde{\tilde{F}}}(t)
\end{aligned}
\tag{33}
$$

where the singular forcing function results from again using eqn 24 to express the effects of the step function turning on $\tilde{\tilde{F}}(t)$. The remaining singular behavior is also the result of using eqn 24 to separate out the singular behavior of the fractional derivatives of acceleration. Again the tilde and double tilde notation identify motion due to previous forces, $\tilde{F}(t)$, and present forces, $\tilde{\tilde{F}}(t)$, respectfully, as in eqns 18 and 19.

Equating the coefficients of the strongest singularities (order $\alpha$) in eqns 32 and 33 yields two equations needed to establish the initial conditions on acceleration.

$$- b\underline{\underline{M}}\,\ddot{\underline{\tilde{u}}}(0^-) - \underline{\underline{k}}_1\underline{\tilde{u}}(0^-) = - b\underline{\tilde{F}}(0^-) \tag{34}$$

$$b\underline{\underline{M}}\,\ddot{\underline{\tilde{\tilde{u}}}}(0^+) + \underline{\underline{k}}_1\,\underline{\tilde{\tilde{u}}}(0^+) = b\underline{\tilde{\tilde{F}}}(0^+) \tag{35}$$

Adding these two equations produces the relationship needed to establish changes in the initial conditions due to stopping and starting of the load histories.

$$\underline{\underline{M}}\left(\ddot{\underline{\tilde{\tilde{u}}}}(0^+) - \ddot{\underline{\tilde{u}}}(0^-)\right) + b^{-1}\underline{\underline{k}}_1\left(\underline{\tilde{\tilde{u}}}(0^+) - \underline{\tilde{u}}(0^-)\right) = \underline{\tilde{\tilde{F}}}(0^+) - \underline{\tilde{F}}(0^-) \tag{36}$$

Since this relationship is based on step loading, which is

incapable of instantaneously changing the displacement or velocity time history between time $0^-$ and $0^+$, one can conclude that

$$\underline{\tilde{\tilde{u}}}(0^+) = \underline{\tilde{u}}(0^-) \tag{37}$$

$$\dot{\underline{\tilde{\tilde{u}}}}(0^+) = \dot{\underline{\tilde{u}}}(0^-) \tag{38}$$

and eqn 36 can now be re-expressed as

$$\ddot{\underline{\tilde{\tilde{u}}}}(0^+) - \ddot{\underline{\tilde{u}}}(0^-) = \underline{\underline{M}}^{-1}\left( \underline{\tilde{\tilde{F}}}(0^+) - \underline{\tilde{F}}(0^-)\right) \tag{39}$$

Thus we see that the change in the initial accelerations is proportional to any instantaneous changes (steps) in the magnitudes of the applied loads at $t = 0$.

To determine the initial accelerations at time $0^+$ one needs to determine the accelerations at time $0^-$ and then add to them the additional component of acceleration from the change in load histories. Should there be a continuous transition from one load history to the other, then

$$\ddot{\underline{\tilde{\tilde{u}}}}(0^+) = \ddot{\underline{\tilde{\tilde{u}}}}(0^-) \tag{40}$$

and the accelerations at time $0^-$ are the accelerations used in the initial value problem. Satisfying the initial conditions on acceleration in this manner effectively removes the $\alpha$ order singular terms on both sides of eqns 32 and 33.

The remaining singular terms in these equations do not have corresponding terms on the force side of the equation. To preserve the equality one must conclude that the coefficients of these singular terms are zero. Note that setting these coefficients to zero, in effect generates the remaining initial

conditions needed in eqn 31. From eqn 32

$$\hat{D}^{(m-2n-\ell)\beta}\left[\underline{\tilde{u}}(0^-)\right] = \underline{0} \qquad \ell = 1,2,3,\cdots,m-2n-1 \tag{41}$$

and from eqn 33

$$\hat{D}^{(m-2n-\ell)}\left[\underline{\tilde{\tilde{u}}}(0^+)\right] = \underline{0} \qquad \ell = 1,2,3,\cdots,m-2n-1. \tag{42}$$

Proof is given in Appendix C. Hence, one can see that the initial values of the fractional derivatives of displacement greater than second order and less than order $\beta m$ must be zero to preserve the equation of motion.

Adding the two equations of motion and recalling that

$$\underline{\tilde{\tilde{u}}}(t) + \underline{\tilde{u}}(t) = \underline{u}(t) \qquad t \geq 0 \tag{43}$$

yields

$$\underline{\underline{M}}(1 + b\hat{D}^{\alpha})\underline{\ddot{u}}(t) + (\underline{\underline{k}}_o + \underline{\underline{k}}_1\hat{D}^{\alpha})\underline{u}(t) = (1+b\hat{D}^{\alpha})\ \underline{\tilde{\tilde{F}}}(t) + \underline{\tilde{G}}(t) \tag{44}$$

which is identical to eqn 8 except for one very important detail: the fractional derivative operator has changed from the original definition, eqn 1, to the modified definition eqn 24. Recall that the modified basis functions use this modified definition as well. In fact the entire initial value problem (constituted by eqns 44, 12, 25, and 31) and its solutions (eqn 30) can be cast in terms of the modified definition of fractional differentiation. Using the composition property, for the modified operator,

$$\hat{D}^{\alpha}[\hat{D}^{\gamma}[u(t)]]=\hat{D}^{\alpha+\gamma}[u(t)], \tag{45}$$

which only applies when the initial values of the fractional derivatives of the displacements are zero, one can straightforwardly demonstrate that eqn 44 leads to a corresponding form of eqn 12 where the $D^{\beta}$ operator is replaced by $\hat{D}^{\beta}$. Similarly, the $D^{\beta}$ operators in eqn 31 can be replaced by $\hat{D}^{\beta}$ as well. Also noting that the particular solution in the initial value problem is in effect an excitation from a quiescent state, one can demonstrate that the first m-1 terms in the series expansion of the kernel in eqn 30 add out when the modal solutions are combined to construct the particular solution. The resulting expression for that part of the modal solutions which describe the system response is given below.

$$y_j(t) \triangleq y_j(0)\, E_{\beta}\left(-(a_j t)^{\beta}\right) + \int_0^t (-a_j^{\beta})^{m-1}\, \hat{D}^{-1-\alpha}\left[E_{\beta}\left(-(a_j t)^{\beta}\right)\right] g_j(t-\tau)\,d\tau \tag{46}$$

Proof is given in Appendix D. The modified equality symbol indicates that this expression is true only in the context of the total system response.

At this point one might be tempted to assert that the original definition of fractional order differentiation, eqn 1, is somehow inappropriate for the initial value problem. Not true. Recall that the initial value problem has insufficient numbers of physically motivated initial values to determine uniquely the overall homogeneous solution as a superposition of solutions to the modified basis equations. The additional auxiliary initial conditions, developed in Appendix C by suppressing singular

behavior at time zero, provided precisely the number of needed initial conditions for a unique solution. In fact the original definition generated this singular behavior without which the initial value problem would flounder for lack of initial information.

To test the robustness of this generalized initial value problem, one needs to ascertain its ability to generate the structural response to impulsive loading. The method entails solving the initial value problem for a step response (using initial accelerations, eqn 39) from a quiescent state and noting that the impulse response is the first derivative of the step response. The structural response for a unit impulse load at the $z^{th}$ degree of freedom of the structure is

$$\underline{u}_{\delta \underline{z}}(t) = \sum_{j=1}^{m \cdot N} \underline{\phi}_{1j} (1+b(-a_j)^q)(-a_j^{\beta})^{2n+q-1} \hat{D}^{-1-\alpha} \left[ E_{\beta}\left( -(a_j t)^{\beta} \right) \right] \underline{\phi}_{1j}^T \underline{z}$$

$$+ \, b \sum_{j=1}^{m \cdot N} \underline{\phi}_{1j} (-a_j)^{2n+q-1} \underline{\phi}_{1j}^T \underline{z} \cdot t \qquad (47)$$

where $\underline{z}$ is an N order column vector of zeroes except the $z^{th}$ element, which is one. Again the solution is seen to be continuous and is expressed in terms of the modified operator and the Mittag-Leffler function. Derivation of this expression is given in Appendix E.

There are several similarities between the initial value problem posed here and the classical initial value problem. First the equations of motion are seen to have both homogeneous and particular solutions which are uniquely dependent on the initial values and the forcing functions respectively. In addition

initial values for all rational order derivatives, up to but not including the highest order in the equation of motion, are needed to determine a unique solution. Also, the impulse response is seen to be a homogeneous solution with special initial conditions. These similarities suggest that other parallels do exist.

Broader Issues in the Generalized Initial Value Problem

The successful construction of the solution to the rational order initial value problem yields some insights into the nature of the underlying mathematics. One of the more unusual results is the number of initial conditions needed to construct a unique solution. It appears that a m/n order equation needs m initial conditions: one condition for the function and a condition for each of the p/n order derivatives $p = 1,2,3,\cdots m-1$. As shown in Appendix C the physically motivated initial conditions – initial displacement, velocity and acceleration – are complemented by precisely the needed number of auxiliary initial conditions. These auxiliary initial conditions are the result of strictly mathematical considerations.

Moreover, how does posing the m/n order equation as a 2m/2n order equation change the nature of the solution for which 2m initial conditions now appear to be needed? In fact the solutions in both cases are identical from which one infers that the basis functions of the two solutions are inter-related.[4] This means that the 1/m order and 1/2m order Mittag-Leffler functions, eqns 28 and 30, are related. These relationships among the rational order Mittag-Leffler functions will be shown later. Choosing to solve the equation as a 2m/2n system versus a m/n system leads to higher order matrix equations and simultaneously provides m more auxiliary conditions ensuring a unique solution. The

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[4] One needs to establish that different but apparently equivalent forms of the equations of motion produce identical solutions. This is similar to posing N second order equations as 2N first order equations where both sets of equations yield the same solution.

requirements for the physically motivated initial conditions remain unchanged.

Assume that the largest common factor for the fractional orders is 1/n where the modified basis equations take the form

$$(\hat{D}^{1/n} + a_j^{1/n})\, y_j(t) = g_j(t), \qquad j = 1,2,3,\cdots,m \tag{48}$$

but one chooses to solve the problem with a basis fraction of 1/2n.

$$(\hat{D}^{1/2n} + a_p^{1/2n}) y_p(t) = g_p(t) \qquad p = 1,2,3,\cdots,2m \tag{49}$$

For each basis equation in eqn 48 with characteristic value $a_j^{1/n}$, there are two corresponding equations in eqn 49 with roots of $\pm\sqrt{-a_j^{1/n}}$, or $\pm i a_j^{1/2n}$. The solutions to these two equations of eqn 49, in the form of eqn 30, are

$$y_p(t) = y_p(0)\; E_{1/2n}\left(-i(a_j t)^{1/2n}\right)$$
$$+ \int_o^t D^{1-1/2n}\left[E_{1/2n}\left(-i(a_j \tau)^{1/2n}\right)\right] g_p(t-\tau)\,d\tau \tag{50}$$

and

$$y_{p+1}(t) = y_{p+1}(0)\; E_{1/2n}\left(+i(a_j t)^{1/2n}\right)$$
$$+ \int_o^t D^{1-1/2n}\left[E_{1/2n}\left(+i(a_j \tau)^{1/2n}\right)\right] g_{p+1}(t-\tau)\,d\tau \tag{51}$$

Solving the problem with a basis fraction of 1/2n instead of 1/n produces m additional auxiliary initial conditions of the form

$$\hat{D}^{p/2n}\, u(0) = 0 \qquad p \text{ odd and } 0 < p < 2m \tag{52}$$

The solutions to eqn 48 also satisfy these initial conditions. In Appendix C it is shown that if $\dot{u}(0)$ exists, then the initial value of any rational order derivative less than one is zero, while u(0) may be independently specified.

The solutions given in eqns 50 and 51 for eqn 49 when combined should produce a corresponding solution to eqn 48, which again based on eqn 30, takes the form

$$y_j(t) = y_j(0) E_{1/n}\left(-(a_j t)^{1/n}\right)$$

$$+ \int_o^t D^{1-1/n}\left[E_{1/n}\left(-(a_j \tau)^{1/n}\right)\right] g_j(t-\tau)\, d\tau \tag{53}$$

However, the solutions in eqns 50 and 51 are not combined arbitrarily. The solutions are combined in a manner consistent with solving eqn 48 as a system of twice as many equations of half the order of differentiation. As such, the loads $g_p(t)$ may be viewed as modal loads generated by $g_j(t)$, and applying the following conditions

$$y_j(0) = y_p(0) + y_{p+1}(0) \tag{54}$$

$$\hat{D}^{1/2n} y_j(0) = 0 \tag{55}$$

the resulting expression for $y_j(t)$ is

$$y_j(t) = y_j(0)\left[\frac{E_{1/2n}\left(i(a_j t)^{1/2n}\right)+E_{1/2n}\left(-i(a_j t)^{1/2n}\right)}{2}\right]$$

(56)

$$+\int_0^t D^{1-1/2n}\left[\frac{E_{1/2n}\left(i(a_j \tau)^{1/2n}\right)-E_{1/2n}\left(-i(a_j \tau)^{1/2n}\right)}{2i}\right]\frac{g_j(t-\tau)}{a_j^{1/2n}}\,d\tau$$

where the integral terms are expressed in terms of eqn 30. However, defining the functions in the brackets as generalized cosine and sine functions

$$\cos_{1/2n}(a_j t) \equiv \frac{E_{1/2n}\left(+i(a_j t)^{1/2n}\right) + E_{1/2n}\left(-i(a_j t)^{1/2n}\right)}{2} \tag{57}$$

$$\sin_{1/2n}(a_j t) \equiv \frac{E_{1/2n}\left(+i(a_j t)^{1/2n}\right) - E_{1/2n}\left(-i(a_j t)^{1/2n}\right)}{2i} \tag{58}$$

greatly simplifies the notation and eqn 56 oecomes

$$y_j(t) = y_j(0)\,\cos_{1/2n}(a_j t) + \int_0^t D^{1-1/2n} a_j^{-1/2n} \sin_{1/2n}(a_j \tau)\, g_j(t-\tau)\,d\tau. \tag{59}$$

The definitions for the generalized sine and cosine functions produce a class of functions having the properties

$$\hat{D}^{1/2n}\sin_{1/2n}(a_j t) = a_j^{1/2n}\cos_{1/2n}(a_j t) \tag{60}$$

$$\hat{D}^{1/2n}\cos_{1/2n}(a_j t) = -a_j^{1/2n}\sin_{1/2n}(a_j t) \tag{61}$$

More general properties of the fractional order sine and cosine functions and their hyperbolic counterparts are given in Appendix F. The appearance of generalized sine and cosine functions can be explained by observing that eqn 48 could have been equivalently posed as a generalized second order differential equation of basis 1/2n.

$$(\hat{D}^{2/2n} + a_j^{\,2/2n})y_j(t) = g_j(t), \tag{62}$$

To complete the development of the relationship between the basis 1/n and 1/2n functions, the composition property, eqn 7, is applied to the derivative operator in the integrand of eqn 59

$$D^{1 - 1/2n}\left[\quad\right] = D^{1 - 1/n + 1/2n}\left[\quad\right] \tag{63}$$

and noting that

$$D^{1/2n}\sin_{1/2n}(a_j t) = a_j^{1/2n}\cos_{1/2n}(a_j t) \tag{64}$$

eqn 59 now becomes

$$y_j(t) = y_j(0)\cos_{1/2n}(a_j t) + \int_0^t D^{1-1/n}\left[\cos_{1/2n}(a_j\tau)\right]g(t-\tau)d\tau \tag{65}$$

Comparing this solution to the solution of eqn 48 given in eqn 53, one concludes that

$$E1/n\left(-(a_j t)^{1/n}\right) = \cos_{1/n}(a_j t)$$

$$= \frac{E_{1/2n}\left(i(a_j t)^{1/2n}\right) + E_{1/2n}\left(-i(a_j t)^{1/2n}\right)}{2} \qquad (66)$$

or

$$2E_{1/n}\left((-a_j t)^{1/n}\right) = E_{1/2n}\left(i(a_j t)^{1/2n}\right) + E_{1/2n}\left(-i(a_j t)^{1/2n}\right) \qquad (67)$$

For $a_j^{1/n}$ negative the relationship becomes

$$2E_{1/n}\left((a_j t)^{1/n}\right) = E_{1/2n}\left((a_j t)^{1/2n}\right) + E_{1/2n}\left(-(a_j t)^{1/2n}\right)$$

$$\equiv 2\cosh_{1/2n}(a_j t) \qquad (68)$$

These relationships can be straightforwardly verified by expanding the Mittag-Leffler functions in terms of their series definition, eqn 26, and adding terms of equal powers. The more general relationship, also verified in the same manner, takes the form

$$E_{m/n}\left(\pm(at)^{m/n}\right) = \frac{1}{p}\sum_{j=1}^{p} E_{m/p}\left((\pm 1)_j^{1/p}(at)^{m/pn}\right) \qquad (69)$$

where $(-1)_j^{1/p}$ are the p different 1/pth roots of minus one. This relationship allows one to demonstrate the equivalence of solutions, u(t), when determined in terms of different basis fractions.

One now concludes that regardless of the choice of

appropriate basis fraction, $\beta$, and the representation of the solution, the uniqueness of the solution may be demonstrated using eqn 69. One also concludes that a change in the basis fraction adjusts the number of auxiliary initial conditions to preserve the uniqueness of the solution, leaving the role of physically motivated initial conditions unaltered. Moreover, the relationships among different order basis function leads to generalized definitions of sine, cosine and the hyperbolic functions (Appendix F). The observation that these generalized functions degenerate into the definitions for the normal sine, cosine, etc., when fractions are set to one, coupled with the observation that eqn 53 becomes the general solution to a first order equation, eqn 62, when n is set to one, lends credence to the view that these fractional order differential equations and their solutions are legitimate generalizations of their integer order counterparts. Furthermore, this solution technique may be viewed as a legitimate generalization of the initial value problem.

Yet another similarity with ordinary differential equations arises when one examines the nature of homogeneous solutions for a fractional order differential equation having repeated roots in its characteristic equation. Posing the differential equation as a system of $\beta$ order modified basis equations in Jordan form (10:35), the two basis equations containing the same root take the form

$$\hat{D}^{\beta} y_j + a_j^{\beta} y_j = 0 \tag{70}$$

$$\hat{D}^{\beta} y_{j+1} + a_j^{\beta} y_{j+1} + y_j = 0 \tag{71}$$

Solving for $y_j$ (eqn 28) and using $y_j$ as a forcing function to determine $y_{j+1}$ (eqn 30) yields

$$y_{j+1}(t) = y_{j+1}(0)\ E_{\beta}\left(-(a_j t)^{\beta}\right) + y_j(0)\ \beta^{-1} t\ D^{1-\beta}\left[E_{\beta}\left(-(a_j t)^{\beta}\right)\right] \tag{72}$$

When $\beta$ is one this solution becomes the solution to a second order differential equation with repeated characteristic values.

$$y_{j+1}(t) = y_{j+1}(0)\ e^{-a_j t} + t\ y_j(0)\ e^{-a_j t} \tag{73}$$

When the root is repeated more than once, $y_{j+1}(t)$ is used to find $y_{j+2}(t)$ and so on until the solution is complete.

$$\hat{D}^{\beta} y_{j+2}(t) + a_j^{\beta} y_{j+2}(t) + y_{j+1} = 0 \tag{74}$$

The general form of the homogeneous solution for m repeated roots is

$$y_{j+m-1}(t) = y_j(0)\ E_{\beta}\left(-(a_j t)^{\beta}\right) + \sum_{k=2}^{m-1} y_{j+k-1}(0)\left[\prod_{p=1}^{k-1}\left((\beta p)^{-1} t D^{1-\beta}\right)\right] E_{\beta}\left(-(a_j t)^{\beta}\right) \tag{75}$$

which for $\beta = 1$ becomes the solution for an ordinary differential equation having constant coefficients with m repeated roots in its

characteristic equation.

When one is confronted with all of the similarities between fractional order and ordinary differential equations, it seems appropriate to examine the fractional order initial value problem as a well posed problem. The existence of solutions has been established. The uniqueness of solutions can be readily established by limiting the class of loads and responses to those having Laplace transforms. One can expand this class of functions to include those loads that produce unique responses through the use of the Green's functions, eqn 46 and 47, and convolution. Continuous dependence on the data can be established using eqns 28, 31, 46, and 47 and noting that the Green's functions are non-singular. In effect the fractional order differential equations exhibit precisely those characteristics expected from ordinary differential equations.

One might ask why not use Laplace transforms to solve the problem and dispense with the formalism constructed here. First, one is not now restricted to those loading and displacement histories having Laplace transforms, although in practice most loadings and responses of engineering interest do have Laplace transforms. In principle it is possible to formulate the initial value problem with Laplace transforms using delays to stop and start loading histories, but several key features of the initial value problem are missed. The transition to the modified operator and the rationale for the formulation of the auxiliary initial conditions can be completely overlooked. Even if the modified operator were discovered using Laplace transforms, the initial values of the fractional derivatives do not appear in the

formulation, because transform theory has correctly set them to zero. The fact that they do not appear obscures their existence and the existence of the auxiliary conditions needed for a unique solution. As a direct consequence, the non-singular nature of the Green's function solutions is missed as well. Also missed are the generalized nature of the basis functions being special Mittag-Leffler functions, the inter-relationships among the different sets of basis functions and the definitions of the generalized sine, cosine, etc., functions spawned by these inter-relationships.

In essence, Laplace transforms by themselves do not readily reveal the structure of the initial value problem, and the needed auxiliary initial conditions for a unique solution are not produced, except possibly in retrospect. As a result the major features of this generalized initial value problem can be missed, and one is never justified in claiming that the class of differ-integral equations solved here are in fact generalized differential equations.

## Conclusions

The most encompassing conclusion is that the class of differ-integral equations, represented by eqn 8, may be viewed as a generalization of ordinary differential equations with constant coefficients. When an equation of motion is posed in terms of m differential equations of fractional order $\beta$, m sets of initial conditions are needed to determine a unique solution. Careful examination of the singular behavior generated by fractional differentiation leads to auxiliary initial conditions that supplement the physically motivated initial conditions producing a total of m sets of initial conditions and a unique solution. The resulting fractional order differential equations are posed in terms of a modified definition of fractional differentiation, which appears to act on the solutions as though they were analytically continued from negative time into positive time. This is precisely the mathematical property needed for a generalized initial value problem.

Several other key features of the mathematical development lend credence to the claim that the mathematics herein embodies a generalization of ordinary differential equations. In every instance, setting the basis fraction to one transforms the solutions into the traditional solutions associated with ordinary differential equations. In addition the solutions to the modified basis equations are posed in terms of Mittag-Leffler functions, long viewed as the fractional order generalization of the exponential function. The general form of the basis solutions contains a portion dependent on the initial condition and a portion dependent the forcing function, clearly identifiable as a

generalized homogeneous solution and a generalized particular solution, respectively. The form of the homogeneous basis solution for the case of repeated characteristic roots is strongly reminiscent of its integer order counterpart, and the two expressions are identical when the basis fraction is one. The generalized sine and cosine functions along with the generalized hyperbolic functions are also very similar to the traditional functions. These similarities lead to generalized identities that also become traditional trigonometric identities when the basis fraction is one. In addition to all these similarities with ordinary differential equations, the observation that the fractional order initial value problem appears to be a well-posed problem makes the case.

This evidence leads one to conclude that the strong similarities between the generalized initial value problem and the traditional initial value problem are not coincidence, and that this generalized initial value problem may be viewed as a legitimate extension of the theory for ordinary linear differential equations with constant coefficients.

## APPENDIX A - Differentiating When Time is Negative

This section presents the technique needed to perform fractional differentiation over the time interval prior to the initial time of the initial value problem. In the process of setting up the initial value problem one needs to examine the mathematical behavior of the system just prior to and just after the initial time ($t = 0$) to determine the nature of additional initial conditions.

Unfortunately, the definition of fractional order differentiation, eqn 1, becomes ambiguous when the lower limit of integration is taken as a negative value.

$$D^{\alpha}\,[y(t)] \equiv \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_{-t_0}^{t}\frac{y(u)}{(t-u)^{\alpha}}\,du \qquad \text{(A-1)}$$

For $t$ less than zero and greater than $-t_o$, the kernel in the integral becomes a complex, multivalued relationship. The definition of fractional differentiation given in eqn 1 does not readily allow one to specify time zero strictly as a matter of convenience. It appears that this definition of fractional differentiation insists that time zero be chosen at times prior to any non-zero values of $y(t)$ for the definition to produce at unique result.

One would like to have the latitude of choosing time zero to coincide with the initial time of the initial value problem. To circumvent the ambiguity raised in eqn A-1, one may employ the two time scales shown in figure 1 on page 15: the $\tilde{t}$ scale to set up the overall problem and the $t$ scale to accommodate the initial value problem. The first time scale, $\tilde{t}$, takes zero to be the

the interval $t_o$) in terms of $\tilde{t}$ produces the equation for prior motion

$$\underline{\underline{M}}\ (1+bD^{\alpha})\ \ddot{\underline{\tilde{u}}}\ (\tilde{t} - t_o) + (\underline{\underline{k}}_o + \underline{\underline{k}}_1\ D^{\alpha})\ \underline{\tilde{u}}\ (\tilde{t} - t_o)$$

$$= (1+bD^{\alpha})\left(\underline{\tilde{F}}(\tilde{t} - t_o) - U(\tilde{t} - t_o)\underline{\tilde{F}}(\tilde{t} - t_o)\right) \tag{A-5}$$

Using eqn A-4 to construct these derivatives, then applying Leibnitz's rule for differentiating the integrals to separate out the singular behavior and substituting t for $\tilde{t} - t_o$ produces eqn 32. Those terms resulting from this process not specifically identified in eqn 32 constitute $\underline{\tilde{G}}(t)$, the pseudo forcing functions, which produces the residual motion, $\underline{\tilde{u}}$,$(t > 0)$ due to previous loading $(t < 0)$ of the structure. This residual motion is superimposed on the response of the structure, $\underline{\tilde{\tilde{u}}}(t)$, to present loading, $\underline{\tilde{\tilde{F}}}(t)$, to produce the total response of the structure for $t > 0$.

onset of any recent history of motion for the structure and the second scale, t, takes zero to be the later time about which one wants to delineate previous and present loading histories. This later time, t, is the time scale of the initial value problem where the initial time is t = 0. The only difference in the time scales is the shift factor $t_o$, the time interval of the recent history of motion. Hence the relationship between the two time scales is

$$t = \tilde{t} - t_o \tag{A-2}$$

Posing eqn A-1 in terms of this time shift yields

$$D^{\alpha}\left[y(\tilde{t}-t_o)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tilde{t}}\int_{-t_0}^{\tilde{t}-t_o}\frac{y(u)}{(\tilde{t}-t_o-u)^{\alpha}}\,du \tag{A-3}$$

and with the change of variable $u = \tau - t_o$ this relationshop becomes

$$D^{\alpha}\left[y(\tilde{t}-t_o)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tilde{t}}\int_{0}^{\tilde{t}}\frac{y(\tau-t_o)}{(\tilde{t}-\tau)^{\alpha}}\,d\tau \tag{A-4}$$

which appears in eqn 17, with $\alpha$ set equal to $\beta$.

The shift of $t_o$ in the time scale restores the single valued property of the fractional derivatives operator.[5] More importantly eqn A-4 allows one to evaluate the behavior of fractional derivatives in the equation of motion, eqn 7, prior to the initial time of the initial value problem which occurs for $\tilde{t} = t_o$. Posing the equation of motion for prior loading (over

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[5] Note that replacing $\tilde{t}$ with t and setting $t_o = 0$ produces the original definition, eqn 1.

APPENDIX B - The Structure of the Eigenvector

This section establishes the systematic structure of the eigenvectors (the columns of the orthonormal transformation matrix in eqn 14) associated with the expanded equations of motion, eqn 12. In particular one demonstrates that the $j^{\underline{th}}$ eigenvector eqn 12 has the following structure

$$\underline{\phi}_j = \begin{Bmatrix} \underline{\phi}_{mj} \\ \vdots \\ \underline{\phi}_{3j} \\ \underline{\phi}_{2j} \\ \underline{\phi}_{1j} \end{Bmatrix} = \begin{Bmatrix} \underline{\phi}_{1j}(-a_j^{\beta})^{m-1} \\ \vdots \\ \underline{\phi}_{1j}(-a_j^{\beta})^{2} \\ \underline{\phi}_{1j}(-a_j^{\beta})^{1} \\ \underline{\phi}_{1j} \end{Bmatrix} \qquad \text{(B-1)}$$

where $\underline{\phi}_{1j}$ is the corresponding $j^{\underline{th}}$ eigenvector associated with the homogeneous form of eqn 11 and $-a_j^{\beta}$ is the eigenvalue associated with both eqns 11 and 12. Stated more succinctly, one will prove that

$$\underline{\phi}_{pj} = \underline{\phi}_{1j}(-a_j^{\beta})^{p-1} \qquad p = 1,2,3,\ldots,m \qquad \text{(B-2)}$$

given the homogeneous form of eqn 12 shown below.

$$(-a_j^{\beta})\left[\begin{array}{c:ccc} \underline{\underline{0}} & & & \\ \vdots & & A & \\ \underline{\underline{0}} & & & \\ \underline{\underline{0}} & & & \\ \hdashline \underline{\underline{c}}_m & \cdots\ \underline{\underline{c}}_3 & \underline{\underline{c}}_2 & \underline{\underline{c}}_1 \end{array}\right] \begin{Bmatrix} \underline{\phi}_{mj} \\ \vdots \\ \underline{\phi}_{3j} \\ \underline{\phi}_{2j} \\ \underline{\phi}_{1j} \end{Bmatrix} + \left[\begin{array}{ccc:c} & & & \underline{\underline{0}} \\ & -A & & \vdots \\ & & & \underline{\underline{0}} \\ & & & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & \cdots\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{c}}_0 \end{array}\right] \begin{Bmatrix} \underline{\phi}_{mj} \\ \vdots \\ \underline{\phi}_{3j} \\ \underline{\phi}_{2j} \\ \underline{\phi}_{1j} \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{Bmatrix} \qquad \text{(B-3)}$$

To begin the proof one focuses on the upper m-1 sets of matrix equations in eqn B-3 and observes that they may be

re-expressed as

$$(-a_j^{\beta})\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{Bmatrix} \underline{\phi}_{m-1\ j} \\ \vdots \\ \underline{\phi}_{2j} \\ \underline{\phi}_{1j} \end{Bmatrix} - \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{Bmatrix} \underline{\phi}_{mj} \\ \vdots \\ \underline{\phi}_{3j} \\ \underline{\phi}_{2j} \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \end{Bmatrix} \quad \text{(B-4)}$$

or

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{Bmatrix} \underline{\phi}_{m-1\ j}(-a_j^{\beta}) - \underline{\phi}_{mj} \\ \vdots \\ \underline{\phi}_{2j}(-a_j^{\beta}) - \underline{\phi}_{3j} \\ \underline{\phi}_{1j}(-a_j^{\beta}) - \underline{\phi}_{2j} \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \end{Bmatrix} \quad \text{(B-5)}$$

The absolute value of the determinant of [A], indicated by $\overline{\det[A]}$, can be expressed as

$$\overline{\det[A]} = \overline{b^{m-1}(\det \underline{\underline{M}})^{m-1}} \quad \text{(B-6)}$$

where $\underline{\underline{M}}$ is the system's mass matrix. Since the mass matrix is positive definite, then the determinant of [A] is always non–zero and [A] is not singular. Hence the only solution to eqn B-5 is the trivial solution, and as a result

$$\begin{aligned} \underline{\phi}_{2j} &= \underline{\phi}_{1j}(-a_j^{\beta}) \\ \underline{\phi}_{3j} &= \underline{\phi}_{2j}(-a_j^{\beta}) \\ &\vdots \\ \underline{\phi}_{mj} &= \underline{\phi}_{m-1\ j}(-a_j^{\beta}) \end{aligned} \quad \text{(B-7)}$$

from which one concludes

$$\underline{\phi}_{pj} = \underline{\phi}_{1j}(-a_j^{\beta})^{p-1} \qquad p = 1,2,3,\ldots,m \quad \text{(B-2)}$$

APPENDIX C - Deriving the Auxiliary Initial Conditions

This section establishes the sources of the auxiliary initial conditions. These auxiliary conditions are needed to determine a unique solution for the initial value problem when the equations of motion are posed as a system of $\beta$ order differential equations.

The auxiliary conditions have two sources. The first is eqn 33, the equation of motion, where certain singular terms appear on the response side of the equation without corresponding singular terms appearing in the applied forces. The equation of motion is preserved by setting the coefficients of these singular terms to zero. These coefficients are the initial values of the fractional derivatives of order greater than 2 and less that $2 + \alpha$ of the structural displacement histories. The source of the remaining auxilary conditions, those derivatives of order between one and two and between zero and one, is the following theorem. Given a set of functions, $\underline{u}(t)$, which are a linear combination of Mittag-Leffler functions as in eqn 31 where $\underline{u}(t)$ have bounded first derivatives at $t = 0$, then all fractional derivatives of these functions of rational order less than one have initial values of zero at $t = 0$.

The proof starts with eqn 32.

$$\underline{u}_h(t) = \sum_{j=1}^{m \cdot N} \underline{\phi}_{1j} \, y_j(t) \qquad (32)$$

where $\underline{\phi}_{1j}$ are the eigenvectors associated with eqn 11 and $y_j(t)$ are Mittag-Leffler homogeneous solutions for the modified basis equations shown in eqn 28. Identifying these homogeneous solutions appearing in eqn 31 with the subscript j produces

$$y_j(t) = y_j(0)\ E_\beta\left(-(a_j t)^\beta\right) \tag{C-1}$$

and using eqn 27 to express these functions in series form produces

$$E_\beta\left(-(a_j t)^\beta\right) \equiv \sum_{p=0}^{\infty} \frac{\left[-(a_j t)^\beta\right]^p}{\Gamma(1+p\beta)} \tag{C-2}$$

Substituting eqn C-2 into eqn C-1 and then substituting the resulting equation for $y_j(t)$ in eqn 32 yields

$$\underline{u}_h(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\ y_j(0) \sum_{p=0}^{\infty} \frac{\left[-(a_j t)^\beta\right]^p}{\Gamma(1+p\beta)} \tag{C-3}$$

Taking the first time derivative of this expression results in

$$\dot{\underline{u}}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\ y_j(0) \sum_{p=1}^{\infty} \frac{(-a_j^\beta)^p\ p\beta\ t^{p\beta-1}}{\Gamma(1+p\beta)} \tag{C-4}$$

where the derivatives of the constant terms in the infinite series (p=0) are now zero. Because one of the two summations has a finite number of terms, the order of summation may be interchanged without loss of generality. Also noting that

$$\frac{p\beta}{\Gamma(1+p\beta)} = \frac{1}{\Gamma(p\beta)} \tag{C-5}$$

the expression for the derviative $\dot{\underline{u}}_h(t)$ becomes

$$\dot{\underline{u}}_h(t) = \sum_{p=1}^{\infty} \frac{t^{p\beta-1}}{\Gamma(p\beta)} \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}(-a_j^\beta)^p\ y_j(0) \tag{C-6}$$

Recall that $\dot{\underline{u}}(0)$ is assumed bounded and at worst has a step

discontinuity at t = 0, and that $\beta$ is of the form 1/n. Therefore, the first n-1 terms of the infinite series must be zero because they are singular and unbounded at t=0. Setting the coefficients of the first n-1 terms equal zero yields

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}(-a_j^{\beta})^{p}\ y_j(0) = \underline{0} \qquad p = 1,2,3,\cdots,n-1 \tag{C-7}$$

Applying the results of Appendix B

$$\underline{\phi}_{kj} = \underline{\phi}_{1j}(-a_j^{\beta})^{k-1} \qquad k = 1,2,3,\cdots,m \tag{B-2}$$

to equation C-7, while noting the m > 2n, results in

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{pj}\ y_j(0) = \underline{0} \qquad p = 1,2,3,\cdots,n-1 \tag{C-8}$$

Examining the structure of the $j^{\underline{th}}$ eigenvector for eqns 12, $\underline{\phi}_j$, shown in equations B-1, and using eqn 31 one concludes that

$$D^{p/n}\ \underline{u}_h(0) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{pj}\ y_j(0) \qquad p = 1,2,3,\cdots,n-1 \tag{C-9}$$

and with eqn C-8, one can further conclude that

$$D^{p/n}\ \underline{u}_h(0) = \underline{0} \qquad p = 1,2,3,\cdots,n-1 \tag{C-10}$$

Because n has not been specified, in principle any n could bechosen. One can conclude that p/n can take on any rational value between zero and one and the proof of the theorem is complete.

However, the theorem need now be applied to determine the initial values of the derivatives of $\underline{u}_h(t)]_{t=0}$ of rational order between one and two. The most intense forces applied to the structure in the initial value problem are the step functions turning off and turning on the loading histories at time zero. A step load is incapable of instantaneously changing the system's kinetic energy and one concludes that the resulting velocity histories are continuous functions for all time. Step loading is also incapable of producing unbounded accelerations, therefore the first derivative of velocity is bounded and piecewise continuous. From this, one can conclude that the velocities $\underline{v}_h(t)$ have bounded derivatives, $\dot{\underline{v}}_h(t)$, and the above theorem applies to the initial values for the fractional derivatives of rational order between zero and one of the velocities. The result is

$$D^{p/n}\ \underline{v}_h(0) = \underline{0} \qquad p = 1,2,3,\cdots,n-1 \qquad \text{(C-11)}$$

Expressed in term of the displacements, these conditions are

$$D^{p/n+1}\underline{u}_h(0) = \underline{0} \qquad p = 1,2,3,\cdots,n-1 \qquad \text{(C-12)}$$

and are seen to be the initial values of the derivatives of rational order between one and two of the displacements.

Given that the initial displacements and the initial velocities are known, coupled with the auxiliary conditions given in eqn C-10 and C-12, the only remaining initial conditions needed in eqn 31 are the initial accelerations and the initial values for the fractional derivatives of $\underline{u}_h(t)$ of rational order between 2 and $2+\alpha$. These conditions are determined by examining the

singular behavior in the equation of motion for previous loading, eqn 32, and the equation of motion for present loading, eqn 33.

The singular behavior in these equations of motion is generated by the $\alpha$ order derivatives of the step functions turning off and turning on the previous and present loading histories and by the $2+\alpha$ order derivatives of the respective resulting dispacement histories. The singular behavior of the forcing function and the $\alpha$ order derivative of the displacements in the equation of motion for present loading, eqn 33, is derived using eqn 23. Calculating the $2+\alpha$ derivative of the displacements is relatively straightforward

$$\underset{\sim}{\tilde{\tilde{u}}}_h(t) = \underset{\sim}{u}_h(0) + \underset{\sim}{\dot{u}}_h(0)\cdot t + \sum_{p=0}^{\infty} \frac{t^{2+p\beta}}{\Gamma(3+p\beta)} \sum_{j=1}^{m\cdot N} \underset{\sim}{\phi}_{1j}(-a_j^{\beta})^{p+2n} y_j(0) \quad \text{(C-13)}$$

The displacements, shown here with the initial displacements and velocities specified and the auxiliary conditions of eqn C-10 and C-12 applied, are first differentiated twice and then successive $\beta$ order derivatives are taken to produce the $2+\alpha$ order derivative. The result is

$$D^{2+\alpha}\, \underset{\sim}{\tilde{\tilde{u}}}_h(t) = \sum_{p=0}^{\infty} \frac{t^{p\beta-\alpha}}{\Gamma(1+p\beta-\alpha)} \sum_{j=1}^{m\cdot N} \underset{\sim}{\phi}_{1j}(-a_j^{\beta})^{p+2n} y_j(0) \quad \text{(C-14)}$$

and recalling the $\beta = 1/n$ and from eqn 10 that $\alpha = \beta q$, this expression becomes

$$D^{2+\alpha}\, \underset{\sim}{\tilde{\tilde{u}}}_h(t) = \sum_{p=0}^{\infty} \frac{t^{(p-q)\beta}}{\Gamma(1-(p-q)\beta)} \sum_{j=1}^{m\cdot N} \underset{\sim}{\phi}_{1j}(-a_j^{\beta})^{p+2n} y_j(0) \quad \text{(C-15)}$$

Separating out the singular terms yields

$$D^{2+\alpha}\, \underline{\tilde{\tilde{u}}}_h(t) = \sum_{p=0}^{q-1} \frac{t^{(p-q)\beta}}{\Gamma(1-(p-q)\beta)} \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\,(-a_j^{\beta})^{p+2n}\, y_j(0) \tag{C-16}$$

$$+ \sum_{p=q}^{\infty} \frac{t^{(p-q)\beta}}{\Gamma(1-(p-q)\beta)} \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\,(-a_j)^{p+2n}\, y_j(0)$$

For $p = 0$ the order of the singular term is $\alpha$ and the coefficient of this singular term

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\,(-a_j^{\beta})^{2n}\, y_j(0) = \underline{\ddot{\tilde{\tilde{u}}}}_h(0^+) \tag{C-17}$$

appears in eqn 44 as the initial value of the acceleration due to the present loading.[6] Recall that eqn 35 resulted from setting equal the coefficients of the $\alpha$ order singular terms on both sides of eqn 33.

The remaining singular terms in eqn C-16 do not have corresponding singular terms in the forces. To preserve the equality of eqn 33 one must conclude that the coefficients of these singular terms are zero. Setting these coefficients to zero yields

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\,(-a_j^{\beta})^{p+2n}\, y_j(0^+) = \underline{0} \qquad p = 1,2,3,\cdots,q-1 \tag{C-18}$$

Again, using the results of Appendix B, eqn B-2, eqn C-18 becomes

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[6] This result can be confirmed by differentiating C-13 twice and setting $t = 0$.

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{\ell j} \; y_j(0^+) = \underline{0} \qquad \ell = 2n+1, 2n+2, 2n+3, \cdots, m-1 \qquad \text{(C-19)}$$

This equation sets the initial values of the fractional derivatives of the displacements of rational order between 2 and $2 + \alpha$ to zero.

Applying the process that led to eqns C-14 through C-19 to the equation of motion for previous loading, eqn 32, and equating the $\alpha$ order singular terms leads to eqn 34. The methods used to determine the fractional derivatives at time $0^-$ are presented in Appendix A. Combining eqns 34 and 35 produces the initial condition on acceleration, eqn 39, and with the initial values of displacements and velocities completes the needed physically motivated initial conditions. These conditions coupled with the auxiliary initial conditions established in eqns C-10, C-12 and C-19 constitute the complete set of initial conditions needed to determine a unique solution for the initial value problem.

APPENDIX D - The Basis Green's Function

The overall particular solution (forced response) for the equations of motion, eqn 8, can be constructed from the solutions of the modified basis equations appearing in eqn 30. The overall solution is a linear combination of the modified basis particular solutions prescribed by the orthonormal transformation for the equations of motion. These linear relationships are, of course, the analogue of eqn 31 applied to the particular solutions.

$$\begin{Bmatrix} (D^{\beta})^{m-1}\underline{u}_p(t) \\ \vdots \\ (D^{\beta})^{2}\underline{u}_p(t) \\ (D^{\beta})^{1}\underline{u}_p(t) \\ \underline{u}_p(t) \end{Bmatrix} = \begin{bmatrix} \underline{\Phi} \end{bmatrix} \begin{Bmatrix} y_{p_{m \cdot N}}(t) \\ \vdots \\ y_{p_3}(t) \\ y_{p_2}(t) \\ y_{p_1}(t) \end{Bmatrix} \tag{D-1}$$

Although one need only employ the lowest set of matrix equations to determine the particular response, this relationship is useful in demonstrating that the overall particular solution does not contain the response generated by the singular terms in the kernel appearing in eqn 30. The kernel is of the form

$$D^{1-\beta}\, E_{\beta}\left(-(a_j t)^{\beta}\right) = \sum_{\ell=1}^{\infty} \frac{(-a_j^{\beta})^{\ell-1}\, t^{\ell\beta-1}}{\Gamma(\ell\beta)} \qquad j = 1,2,3,\cdots m\cdot N \tag{D-2}$$

These kernels are the particular solutions when the forcing functions, $g_j(t)$, are unit impulse functions.

To examine the behavior of the impulse response, one calculates first the response to a unit step load and then takes the first derivative of the step response to produce the impulse response. The response of the $j^{\underline{th}}$ modified basis equation for a

unit step applied at the $z^{\underline{th}}$ degree of freedom is

$$y_{p_{jz}}(t) = \int_0^t D^{1-\beta} \, E_\beta\left(-(a_j t)^\beta\right) \, g_{jz}(t-\tau) \, d\tau \qquad \text{(D-3)}$$

where

$$g_{jz}(t) = \underline{\phi}_{1j}^T \, \underline{z} \, U(t) \qquad \text{(D-4)}$$

Here $\underline{\phi}_{1j}^T$ is the transpose of the $j^{\underline{th}}$ eigenvector of eqn 11, $\underline{z}$ is a column matrix of zero elements except the $z^{\underline{th}}$ which is one and $U(t)$ is the unit step function. Substituting eqn D-4 into eqn D-3 and evaluating the integral produces the step response, $y_{s_{jz}}(t)$.

$$y_{s_{jz}}(t) = I^\beta\left[E_\beta\left(-(a_j t)^\beta\right)\right] \underline{\phi}_{1j}^T \, \underline{z} \qquad \text{(D-5)}$$

$I^\beta$ is the fractional integral of order $\beta$ and is defined as

$$I^\beta \, [y(t)] = \frac{1}{\Gamma(\beta)} \int_0^t \frac{y(t-\tau)}{(\tau)^{1-\beta}} \, d\tau \qquad \text{(D-6)}$$

The $\beta$ order integral of the $\beta$ order Mittag-Leffler function appearing in equation D-5 can be shown to be

$$I^\beta \left[E_\beta\left(-(a_j t)^\beta\right)\right] = \sum_{\ell=1}^{\infty} \frac{(-a_j^\beta)^{\ell-1} t^{\ell\beta}}{\Gamma(1+\ell\beta)} \qquad \text{(D-7)}$$

Using eqns D-5 and D-7 and summing all the step responses of

the basis functions using the lowest set of matrix equations in D-1 produces the structure's step response.

$$\underline{u}_{s_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^T\, \underline{z} \sum_{\ell=1}^{\infty} \frac{(-a_j^{\beta})^{\ell-1} t^{\ell\beta}}{\Gamma(1+\ell\beta)} \tag{D-8}$$

However, noting that all the basis step responses, eqn D-5, are zero at $t = 0$ (see eqn D-7), one can conclude from eqn D-1 that all of the fractional derivatives

$$D^{\beta k}\underline{u}_{s_z}(t)\Big]_{t=0} = \underline{0} \qquad k = 0,1,2,3,\cdots,m-1 \tag{D-9}$$

of the step response are zero at $t = 0$ as well. In other words, if all the step responses are zero at $t = 0$, then all of the elements in the right column vector of eqn D-1 are zero for $t = 0$ and eqn D-9 follows.

It follows from applying eqn D-9 to eqn D-8 that the first m-1 terms in the time series representation of $\underline{u}_{s_z}(t)$, eqn D-8, must also be zero. The resulting expression for the step response is

$$\underline{u}_{s_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^T\, \underline{z} \sum_{\ell=m}^{\infty} \frac{(-a_j^{\beta})^{\ell-1} t^{\ell\beta}}{\Gamma(1+\ell\beta)} \tag{D-10}$$

or

$$\underline{u}_{s_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^T\, \underline{z} \sum_{\ell=0}^{\infty} \frac{(-a_j^{\beta})^{2n+q+\ell-1}\, t^{\beta(2n+q+\ell)}}{\Gamma(1+\beta(2n+q+\ell))} \tag{D-11}$$

Taking the first derivative of $\underline{u}_{s_z}$ produces the impulse response,

$\underline{u}_{\sigma_z}(t)$. Recall that $\beta = 1/n$.

$$\underline{u}_{\delta_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^T\, \underline{z} \sum_{\ell=0}^{\infty} \frac{(-a_j^{\beta})^{2n+q+\ell-1}\, t^{1+\alpha+\ell\beta}}{\Gamma(2+\alpha+\ell\beta)} \qquad \text{(D-12)}$$

However, the overall impulse response may also be expressed as a linear combination of the effective impulse responses associated with each mode as shown below.

$$\underline{u}_{\delta_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\, y_{\delta_{jq}}(t) \qquad \text{(D-13)}$$

Using eqns D-12 and D-13 one can determine the expression for the effective impulse response associated with each mode, and it is seen to be

$$y_{\delta_{jz}}(t) = (-a_j^{\beta})^{2n+q-1} \sum_{\ell=0}^{\infty} \frac{(-a_j^{\beta})^{\ell}\, t^{1+\alpha+\ell\beta}}{\Gamma(2+\alpha+\ell\beta)}\; \underline{\phi}_{1j}^T\, \underline{z} \qquad \text{(D-14)}$$

Given the effective modal impulse response one can now work back to determine the new form for the kernel of the convolution integral in the particular solution. The new kernel is

$$\sum_{\ell=m}^{\infty} \frac{(-a_j^{\beta})^{\ell-1}\, t^{\ell\beta-1}}{\Gamma(\ell\beta)} = (-a_j^{\beta})^{m-1}\, \hat{D}^{-1-\alpha} \left[E_{\beta}\left(-(a_j t)^{\beta}\right)\right] \qquad \text{(D-15)}$$

The first term in the kernel is order $1+\alpha$. The earlier expression for the kernel, eqn D-2, shows a first term of order $\beta-1$. The difference in these expressions is based on the observation that

eqn D-9 demonstrates that the first m-1 terms of the basis step responses sum to zero when constructing the overall step response using eqn D-1. Consequently, the first m-1 terms of the basis impulse responses sum to zero when constructing the overall impulse response. Without loss of generality these m-1 terms may be left out of the kernel when calculating the overall particular solution for any loading history. The general form of the overall particular solution is given in eqn 46.

## APPENDIX E - The Response to Impulsive Loading

In Appendix D the impulse response or Green's function was based on $g_j(t)$, eqn 22, being impulsive where $g_j(t)$ is a combination of the pseudo forces and the stress operator from the viscoelastic model acting on the applied forces. This appendix presents the general form of the response of the structure to an impulsive force. The derivation is based on the step response for which the foundation has been laid in eqns 32 through 44 and Appendices A, B, and C. Paralleling Appendix D, the determination of the impulse response rests on the observation that in all linear systems the first derivative of the step response is the impulse response.

The derivation of the step response is based on solving eqn 44. Note that for step loading with no previous motion, $\underline{\tilde{G}}(t)$ and $\hat{D}^{\alpha}\underline{\tilde{\tilde{F}}}(t)$ are zero in eqn 44. Casting the resulting equation of motion in expanded format, eqn 12, allows one to solve for the step response by specifying the initial accelerations in the state vector. The expressions for the initial accelerations based on eqn 39 is

$$\ddot{\underline{\tilde{\tilde{u}}}}(0^+) = \underline{\underline{M}}^{-1}\underline{\tilde{\tilde{F}}}(0^+) \tag{E-1}$$

where the tildes will be dropped because there is no longer a need to distinguish between previous and present loading and responses.

The remaining force vector in eqn 44, $\underline{\tilde{\tilde{F}}}(t)$, is taken to be a unit step load applied at the $z^{\underline{th}}$ degree of the freedom. This load vector $\underline{z}$ is all zeros except for the $z^{\underline{th}}$ element, which is one. The resulting expression for the initial accelerations is

$$\underline{\ddot{u}}(0^+) = \underline{\underline{M}}^{-1}\underline{z} \tag{E-2}$$

Note that the step force generates both homogeneous and a particular parts in the response. The homogeneous part arises from the initial accelerations and the particular part arises from the force term $\underline{\tilde{\tilde{F}}}(t)$ in the equation of motion. For simplicity, the particular part of the step response will not be introduced until after the homogeneous solution is determined. The particular solution and its derivatives of order up to and including $2+\alpha-\beta$ are initially zero and have no effect on the form of the homogeneous part.

Based on eqn 31 the initial values of the homogeneous parts of the basis functions should satisfy

$$\begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{\underline{M}}^{-1}\underline{z} \\ \underline{0} \\ \vdots \\ \underline{0} \end{Bmatrix} = \begin{bmatrix} \underline{\Phi} \end{bmatrix} \begin{Bmatrix} y_{m \cdot N}(0) \\ \vdots \\ y_3(0) \\ y_2(0) \\ y_1(0) \end{Bmatrix} \tag{E-3}$$

Here $\underline{\underline{M}}^{-1}\underline{z}$ is the prescribed initial accelerations and is the $2n + 1^{\underline{st}}$ column vector from the bottom of the array. To determine the homogeneous part one needs to solve for $y_j(0)$, $j = 1,2,3,\cdots, m \cdot N$. The orthonormal transformation matrix for the expanded equations of motion, eqn 12, has the property

$$\begin{bmatrix} \underline{\Phi} \end{bmatrix}^T \begin{bmatrix} M^* \end{bmatrix} \begin{bmatrix} \underline{\Phi} \end{bmatrix} = \begin{bmatrix} \diagdown_{I} \diagdown \end{bmatrix} \tag{E-4}$$

where $[M^*]$ is the pseudo mass matrix associated with the expanded equations of motion, eqn 12. Hence premultiplying both sides of equation E-3 by $[\Phi]^T[M^*]$, which is $[\Phi]^{-1}$, produces

$$\begin{bmatrix} \underline{\Phi} \end{bmatrix} \begin{bmatrix} M^* \end{bmatrix} \begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{\underline{M}}^{-1}\underline{z} \\ \underline{0} \\ \vdots \\ \underline{0} \end{Bmatrix} = \begin{Bmatrix} y_{m\cdot N}(0) \\ \vdots \\ y_3(0) \\ y_2(0) \\ y_1(0) \end{Bmatrix} \tag{E-5}$$

Due to the systematic structure of $[M^*]$ and the eigenvectors associated with the expanded equations of motion shown in Appendix B, and the location of $\underline{\underline{M}}^{-1}\underline{z}$ in the array (the $2n+1^{\underline{st}}$ column vector); it follows that E-5 reduces to

$$b \begin{Bmatrix} \underline{\phi}_{qm\cdot N}^T \underline{z} \\ \vdots \\ \underline{\phi}_{q3}^T \underline{z} \\ \underline{\phi}_{q2}^T \underline{z} \\ \underline{\phi}_{q1} \underline{z} \end{Bmatrix} = \begin{Bmatrix} y_{m\cdot N}(0) \\ \vdots \\ y_3(0) \\ y_2(0) \\ y_1(0) \end{Bmatrix} \tag{E-6}$$

from which one concludes

$$y_j(0) = b\ \underline{\phi}_{qj}^T\ \underline{z} \qquad j = 1,2,3,\cdots,m\cdot N \tag{E-7}$$

where q is defined in eqn 10. Using eqn B-2, this expression for the initial values of the basis functions becomes

$$y_j(0) = b\ \underline{\phi}_{1j}^T\ (-a_j^{\beta})^{q-1}\ \underline{z} \qquad j = 1,2,3,\cdots,m\cdot N \tag{E-8}$$

The initial values, $y_j(0)$, are now uniquely determined;

however, certain conditions resulting from eqn E-8 will be useful in simplifying the expressions for the homogeneous part. Using eqn E-8 to substitute for the initial values in eqn E-3 yields

$$\begin{Bmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{\underline{M}}^{-1}\underline{z} \\ \underline{0} \\ \vdots \\ \underline{0} \end{Bmatrix} = \begin{bmatrix} \underline{\Phi} \end{bmatrix} \begin{Bmatrix} (-a_{m\cdot N}^{\beta})^{q-1} \\ \vdots \\ (-a_3^{\beta})^{q-1} \\ (-a_2^{\beta})^{q-1} \\ (-a_1^{\beta})^{q-1} \end{Bmatrix} b\, \underline{\phi}_{1j}^{T}\, \underline{z} \tag{E-9}$$

and as a result one sees that

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{pj}\, (-a_j^{\beta})^{q-1} = \underline{0} \qquad p = 1,2,3,\cdots,m-1 \quad p \neq 2n+1 \tag{E-10}$$

and again using eqn B-2, eqn E-10 becomes

$$\sum_{j=1}^{m\cdot N} \underline{\phi}_{1j} (-a_j^{\beta})^{p+q-2} = \underline{0} \qquad p = 1,2,3,\cdots,m-1 \quad p \neq 2n+1 \tag{E-11}$$

These conditions will be applied to the homogeneous part of the response.

Based on eqn 31, the homogeneous part of the step response is

$$\underline{u}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\; y_j(0)\; E_{\beta}[-(a_j t)^{\beta}] \tag{E-12}$$

Using eqn E-8 to insert the initial values and presenting the Mittag-Leffler functions in terms of their series representations produces the homogeneous part of the step response, $\underline{u}_h(t)$.

$$\underline{u}_{h_z}(t) = \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j}\; b\; (-a_j^{\beta})^{q-1}\; \underline{\phi}_{1j}^{T}\; \underline{z} \sum_{\ell=0}^{\infty} \frac{(-a_j^{\beta})^{\ell} t^{\ell\beta}}{\Gamma(1+\ell\beta)} \tag{E-13}$$

Interchanging the order of summation and re-arranging terms results in

$$\underline{u}_{h_z}(t) = \sum_{\ell=0}^{\infty} \frac{t^{\ell\beta}}{\Gamma(1 + \ell\beta)} \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, (-a_j^{\beta})^{\ell+q-1} \, \underline{\phi}_{1j}^T \, \underline{z} \, . \qquad \text{(E-14)}$$

This form of the response allows one to apply, in a straightforward manner, the conditions given in eqn E-11, which removes several of the lower order terms in the time series. The resulting form of the homogeneous part of the step response is

$$\underline{u}_{h_z}(t) = \frac{t^2}{2} \cdot \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, (-a_j^{\beta})^{2n+q-1} \underline{\phi}_{1j}^T \, \underline{z}$$

$$+ \sum_{\ell=2n+q}^{\infty} \frac{t^{\ell\beta}}{\Gamma(1+\ell\beta)} \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, (-a_j^{\beta})^{\ell+q-1} \underline{\phi}_{1j}^T \, \underline{z} \qquad \text{(E-15)}$$

Having applied all the initial conditions and constructed the homogeneous part of step response, one takes the first derivative of eqn E-15. Recall that $(2n+q)\beta$ is equal to $2+\alpha$.

$$\dot{\underline{u}}_{h_z}(t) = \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, (-a_j^{\beta})^{2n+q-1} \underline{\phi}_{1j}^T \, \underline{z} \cdot t$$

$$+ \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, (-a_j^{\beta})^{2n+2q-1} \, \underline{\phi}_{1j} \, \underline{z} \sum_{\ell=0}^{\infty} \frac{t^{(n+q+\ell)\beta} (-a_j^{\beta})^{\ell}}{\Gamma(1+n+q+\ell)} \qquad \text{(E-16)}$$

Using more compact notation eqn E-16 becomes

$$\dot{\underline{u}}_{h_z}(t) = \sum_{j=1}^{m \cdot N} b \, \underline{\phi}_{1j} \, \underline{\phi}_{1j}^T \, \underline{z} \left\{ (-a_j^{\beta})^{2n+q-1} t \right.$$

$$\left. + (-a_j^{\beta})^{2n+2q-1} \hat{D}^{-1-\alpha} \left( E_{\beta} \left( -(a_j t)^{\beta} \right) \right) \right\} \qquad \text{(E-17)}$$

This equation is the expression for the part of the impulse response dependent on the initially induced velocities (or momentum). Recall however, that particular part of the step response, based on eqns 46 and D-4,

$$\underline{u}_{p_z}(t) = \sum_{j=1}^{m \cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^{T}\, \underline{\underline{z}}\, (-a_j^{\beta})^{m-1}\, \hat{D}^{-2-\alpha} \left[ E_{\beta}(-(a_j t)^{\beta}) \right] \tag{E-18}$$

is not yet included. Taking its first derivative

$$\dot{\underline{u}}_{p_z}(t) = \sum_{j=1}^{m \cdot N} \underline{\phi}_{1j}\, \underline{\phi}_{1j}^{T}\, \underline{\underline{z}}\, (-a_j^{\beta})^{m-1}\, \hat{D}^{-1-\alpha} \left[ E_{\beta}(-(a_j t)^{\beta}) \right] \tag{E-19}$$

and combining it with eqn E-17 yields eqn 47.

APPENDIX F - A Class of Generalized Functions

Oldham and Spanier (17:122) present the one-half order analogue of the exponential function. This function is predicated on the definition of fractional differentiation given in eqn 1, and the function exhibits singular behavior at $t = 0$. Note that the formulation of the fractional order initial value problem led to the exclusive adoption of the modified definition of fractional order differentiation, $\hat{D}[y(t)]$ defined in eqn 23. As a result one is motivated to construct a set of generalized functions based on the modified definition of fractional order differentiation. As expected the singular behavior does not appear in the derivatives of the modified analogues. The absences of the singular behavior lead to a modified set of generalized functions having properties remarkably similar to their common counterparts. Moreover, this set of modified functions is precisely those functions found in the solutions to the modified basis equations which are the foundation of the fractional order initial value problem.

This class of functions is built around the special Mittag-Leffler function

$$E_\beta\left[-(at)^\beta\right] = \sum_{p=0}^{\infty} \frac{(-(at)^\beta)^p}{\Gamma(1+p\beta)} \qquad \text{(F-1)}$$

and the modified definition of fractional differentiation.

$$\hat{D}^\beta[u(t)] \equiv \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'(t-\tau)}{\tau^\beta}\, d\tau \qquad 0 < \beta < 1 \qquad \text{(F-2)}$$

The special Mittag-Leffler function has the property

$$\hat{D}^{\beta}\left[E_{\beta}\left(-(at)^{\beta}\right)\right] = \left(-a^{\beta}\right)E_{\beta}\left(-(at)^{\beta}\right) \qquad \text{(F-3)}$$

which is of course analogous to

$$\frac{d}{dt}\, e^{-at} = -\, ae^{-at} \qquad \text{(F-4)}$$

In fact eqn F-3 becomes eqn F-4 when $\beta$ is set to one.(13:527) Consequently, for our purposes the special Mittag-Leffler function, eqn F-1, is taken to be the generalized fractional order exponential function. The property given in F-3 may be proven by taking the $\beta$ order derivative, eqn F-2, of each term in the series appearing in eqn F-1.

Having established the fractional order exponential function, the definitions of the fractional order sine and cosine functions take the form

$$\sin_{\beta}\left((at)^{\beta}\right) \equiv \frac{E_{\beta}\left[i(at)^{\beta}\right] - E_{\beta}\left[-i(at)^{\beta}\right]}{2i} \qquad \text{(F-5)}$$

$$\cos_{\beta}\left((at)^{\beta}\right) \equiv \frac{E_{\beta}\left[i(at)^{\beta}\right] + E_{\beta}\left[-i(at)^{\beta}\right]}{2} \qquad \text{(F-6)}$$

These two functions have derivatives very similar to the regular sine and cosine functions

$$\hat{D}^{\beta}\left[\sin_{\beta}\left((at)^{\beta}\right)\right] = a^{\beta}\cos_{\beta}\left((at)^{\beta}\right) \qquad \text{(F-7)}$$

$$\hat{D}^{\beta}\left[\cos_{\beta}\left((at)^{\beta}\right)\right] = -a^{\beta}\sin_{\beta}\left((at^{\beta})\right) \qquad \text{(F-8)}$$

Note that setting $\beta$ to one in the above four equations yields the exponential representation of the normal sine and cosine functions as well as the properties of their derivatives. Using eqns F-5 and F-6 one can straightforwardly demonstrate that

$$E_\beta\left(\pm i(at)^\beta\right) = \cos_\beta\left((at)^\beta\right) \pm i \sin_\beta\left((at)^\beta\right) \tag{F-9}$$

which is the generalized form of Euler's formula. This relationship also leads to

$$E_\beta\left[i(at)^\beta\right] \cdot E_\beta\left[-i(at)^\beta\right] = \cos^2_\beta\left((at)^\beta\right) + \sin^2_\beta\left((at)^\beta\right) \tag{F-10}$$

which for $\beta$ set to one produces

$$1 = \cos^2(at) + \sin^2(at) \tag{F-11}$$

The definitions for the fractional order hyperbolic sine and cosine functions are

$$\sinh_\beta\left((at)^\beta\right) = \frac{E_\beta\left((at)^\beta\right) - E_\beta\left(-(at)^\beta\right)}{2} \tag{F-12}$$

$$\cosh_\beta\left((at)^\beta\right) = \frac{E_\beta\left((at)^\beta\right) + E_\beta\left(-(at)^\beta\right)}{2} \tag{F-13}$$

where their derivatives are seen to be

$$\hat{D}^\beta\left[\sinh_\beta\left((at)^\beta\right)\right] = a^\beta \cosh_\beta\left((at)^\beta\right) \tag{F-14}$$

$$\hat{D}^\beta\left[\cosh_\beta\left((at)^\beta\right)\right] = a^\beta \sinh_\beta\left((at)^\beta\right) \tag{F-15}$$

These functions also satisfy

$$E_\beta\left(\pm(at)^\beta\right) = \cosh_\beta\left((at)^\beta\right) \pm \sinh_\beta\left((at)^\beta\right) \qquad \text{(F-16)}$$

and

$$E_\beta\left[(at)^\beta\right] E_\beta\left[-(at)^\beta\right] = \cosh^2_\beta\left((at)^\beta\right) - \sinh^2_\beta\left((at)^\beta\right) \qquad \text{(F-17)}$$

All of these definitions and relationships for the generalized hyperbolic functions are seen to reduce to traditional definitions and relationships when $\beta$ is one.

The remarkable similarity between the behavior of these generalized functions and their ordinary counterparts, not exhibited by Oldham's generalized functions, is directly attributable to the absence of the singular behavior in Oldham's functions.

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